Some bounds on alliances in trees

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Abstract

Given a simple graph $G = (V, E)$, a subset S of the vertices is called a *global defensive alliance* if S is a dominating set and for every vertex v in S at least half of the vertices in the closed neighborhood of v are in S. Similarly, a subset S is called a *global offensive alliance* if S is a dominating set and for every vertex v not in S at least half of the vertices in the closed neighborhood of v are in S. In this paper, we study the minimum cardinality global defensive and global offensive alliances of complete k-ary trees. We also give bounds on the difference between these two parameters for general trees.

Keywords: Global alliances, offensive alliance, defensive alliance, complete k-ary trees.

1 Introduction

The study of alliances in graphs was first introduced by Hedetniemi, Hedetniemi and Kristiansen [5]. They introduced the concepts of defensive and offensive alliances, global offensive and global defensive alliances and studied alliance numbers of a class of graphs such as cycles, wheels, grids and complete graphs. Haynes et al. [3] studied the global defensive alliance numbers of different classes of graphs. They gave lower bounds for general graphs, bipartite graphs and trees, and upper bounds for general graphs and trees.

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Rodriquez-Velazquez and Sigarreta [9] studied the defensive alliance number and the global defensive alliance number of line graphs. A characterization of trees with equal domination and global strong defensive alliance numbers was given by Haynes, Hedetniemi and Henning [4]. Rodriguez-Velazquez and Sigarreta [6] gave bounds for the defensive, offensive, global defensive, global offensive alliance numbers in terms of the algebraic connectivity, the spectral radius, and the Laplacian spectral radius of a graph. They also gave bounds on the global offensive alliance number of cubic graphs in [7] and the global offensive alliance number for general graphs in [8].

Balakrishnan et al. [1] studied the complexity of global alliances. They showed that the decision problems for global defensive and global offensive alliances are both NP-complete for general graphs.

Given a simple graph $G = (V, E)$ and a vertex $v \in V$, the *open neighborhood* of v, $N(v)$, is defined as $N(v) = \{u : uv \in E\}$. The *closed neighborhood* of v, denoted by $N[v]$, is $N[v] = N(v) \cup \{v\}$. Given a set $X \subset V$, the *boundary* of X, denoted by $\delta(X)$, is the set of vertices in $V - X$ that are adjacent to at least one member of X. A set $X \subset V$ is called a *dominating set* if $\delta(X) = V - X$. The *subgraph induced by* X, denoted by $< X >$, is the graph with vertex set X and edge set $E(X)$ where $uv \in E(X)$ if and only if $uv \in E(G)$.

Definition 1.1. *A set $S \subset V$ is a* defensive alliance *if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. A defensive alliance S is called a* global defensive alliance *if S is also a dominating set.*

Definition 1.2. *A set $S \subset V$ is an* offensive alliance *if for every $v \in \delta(S)$, $|N[v] \cap S| \geq |N[v] - S|$. An offensive alliance S is called a* global offensive alliance *if S is also a dominating set.*

Definition 1.3. *The* global defensive(offensive) alliance number *of G is the cardinality of a minimum size global defensive(offensive) alliance in G, and is denoted by $\gamma_a(G)(\gamma_o(G))$. A minimum size global defensive(offensive) alliance is called a $\gamma_a(G)$-set ($\gamma_o(G)$-set).*

In this paper, we study the global defensive and global offensive alliance numbers of trees. We find the asymptotic order of global defensive alliance number of complete k-ary trees, and compute exactly the global offensive alliance number. We also give a sharp bound on the difference between the global offensive and global defensive alliance numbers for a general tree. The results of the paper were first reported in [2] without any proofs. In this paper, we provide the complete proofs.

The rest of the paper is organized as follows. In Section 2, we find the global defensive alliance number of complete binary and complete ternary trees. We also find lower and upper bounds for the global defensive alliance number of complete k-ary trees, and determine the asymptotic order. In Section 3, we find the global offensive alliance number of complete k-ary trees. In Section 4, we compare the global offensive and global defensive alliance numbers of a general tree, giving a sharp bound on the difference.

2 Defensive Alliances in Complete k-ary Trees

A *k-ary tree* is a rooted tree where each node has at most k children. The *depth* of a vertex v is the (edge) length of the shortest path from the root to v. The *depth of a tree* is the maximum depth of a vertex. We denote by L_i the set of vertices that have depth i in a rooted tree. A *complete k-ary tree* is a k-ary tree in which all the leaves have the same depth and all the nodes except the leaves have k children. We let $T_{k,d}$ be the complete k-ary tree with depth d. $T_{2,d}$ is called the *complete binary tree* and in the sequel will be denoted by T_d. In this section, we compute $\gamma_a(T_{k,d})$ exactly for $k = 2$ and $k = 3$, and give close lower and upper bounds for general k.

We label the vertices of $T_{k,d}$ as follows: let (i, j) be the j^{th} vertex from the left at depth i, for $(0 \leq i \leq d)$, $(1 \leq j \leq k^i)$. For example, the vertex $(0, 1)$ is the root and the vertex $(d, 1)$ is the leftmost leaf of the tree. The parent of the node (i, j) is $(i - 1, \lceil \frac{j}{k} \rceil)$. We first find the global defensive alliance number of the complete binary tree.

Theorem 2.1. *Let n be the order of the complete binary tree T_d. Then $\gamma_a(T_d) = \lceil \frac{2}{5} n \rceil$ for any d.*

Proof. Obviously, if $d = 1$ then we have $n = 3$ and exactly 2 of the vertices must be in the alliance. Therefore, assume that $d \geq 2$.

For $j < i$, we define

$$S_{i,j} = \cup_{k=j}^{i} L_k.$$

Consider the set $S_{d,d-2}$. It induces a forest where each component is a copy of T_2. Given a $\gamma_a(T_d)$-set R, we claim that each component of $< S_{d,d-2} >$ contains at least 3 vertices of R. Moreover, we claim that there is a unique way that R can contain exactly 3 vertices in this component.

Without loss of generality, consider the component which is the subtree rooted at vertex $(d-2, 1)$. Observe that if a vertex $v \in R$ is not a leaf, then at least one of its neighbors is also in R. If $(d - 1, 1) \notin R$, then both of its children, $(d, 1)$ and $(d, 2)$, must be in R because otherwise the alliance is not global. Similarly, if the vertex $(d-1,2) \notin R$, then both of its children, $(d,3)$ and $(d,4)$, must be in R. Therefore, without loss of generality suppose that $(d-1,1)$ is in R. At least one of the neighbors of $(d-1,1)$ must be in R. Whichever neighbor of $(d-1,1)$ is in R, it does not dominate $(d,3)$ and $(d,4)$. Thus, in the subtree rooted at $(d-2,1)$ we need at least 3 vertices to be in R. In fact, there is only one way of choosing these 3 vertices if R is to be a global defensive alliance. This happens when we take $(d-2,1)$, $(d-1,1)$ and $(d-1,2)$ to be in R. So we have shown that every subtree of T_d rooted at a node at depth $d-2$ must contain at least 3 nodes in any global defensive R, and R can contain exactly 3 vertices in a unique way.

Next, we consider $S_{i,i-3}$, $i \geq 3$. This set induces a graph which is a forest. All the components of $< S_{i,i-3} >$ are isomorphic copies of the tree T_3. We claim that each component C contains at least 3 vertices of R. Moreover, there is a unique way that R can contain exactly 3 vertices of C. Without loss of generality, consider the component C_1 which contains the vertex $(i-3,1)$. For each of the vertices $(i-1,1),(i-1,2),(i-1,3),(i-1,4)$ either it or one of its neighbors must be in R since otherwise R is not a dominating set. Furthermore, each such vertex in R must also have a neighbor in R. The best case we have is when we take $(i-2,1),(i-2,2)$ to be in R. But each of these vertices need to have a neighbor in R. So we must at least include $(i-3,1)$ in R as well. It is easy to see that if R contains exactly 3 vertices in C_1 then this is the unique combination.

Next, we define a set $S^{'}$ which is a global defensive alliance and show that $S^{'}$ contains exactly 3 vertices in all the connected components of $< S_{d,d-2} >$ and $< S_{i-3,i} >$, for certain i's. First let S be the set

$$S = \cup_{i=0}^{\lfloor (d-2)/4 \rfloor} S_{d-4i-1,d-4i-2}.$$

We consider two cases.

Case 1: $d \neq 1 \bmod 4$.

Subcase (i): $d \equiv 2 \bmod 4$.

Then $S = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{1,0}$. Then it is easy to see that $S^{'} = S$ is a global defensive alliance. Also, $S^{'}$ contains exactly 3 vertices of each component of $< S_{d-1,d-2} >$ and $< S_{d-4i-1,d-4i-2} >$ for all $0 \leq i \leq (d-2)/4$. Therefore, by what we proved above, $S^{'}$ is a global defensive alliance of minimum cardinality. The size of $S^{'}$ is

$$\begin{aligned}
|S^{'}| &= (2^{d-1}+2^{d-2}+2^{d-5}+2^{d-6}+2^{d-9}+2^{d-10}+...+2^{1}+2^{0}) \\
&= 3(2^{d-2}+2^{d-6}+2^{d-10}+...+2^{0}) \\
&= 3\left(\frac{((2^{4})^{\frac{d+2}{4}}-1)}{2^{4}-1}\right) = \frac{1}{5}(2^{d+2}-1) \\
&= \frac{1}{5}(2(n+1)-1) = \frac{2}{5}n+\frac{1}{5}.
\end{aligned}$$

Thus, if $d \equiv 2 \bmod 4$, then $\gamma_a(T_d) = \frac{2}{5}n + \frac{1}{5}$.

Subcase (ii): $d \equiv 3 \bmod 4$.

Then $S = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{2,1}$. Clearly, $S^{'} = S$ is a global defensive alliance. By the same argument as in subcase (i), $S^{'}$ contains the least number of vertices any $\gamma_a(T_d)$-set must contain. Therefore, $S^{'}$ is a $\gamma_a(T_d)$-set. The size of $S^{'}$ is

$$\begin{aligned}
|S^{'}| &= (2^{d-1}+2^{d-2}+2^{d-5}+2^{d-6}+2^{d-9}+2^{d-10}+...+2^{2}+2^{1}) \\
&= 3(2^{d-2}+2^{d-6}+2^{d-10}+...+2) \\
&= 3\left(\frac{2((2^{4})^{\frac{d+1}{4}}-1)}{2^{4}-1}\right) \\
&= \frac{1}{5}(2(2^{d+1}-1)) = \frac{2}{5}n.
\end{aligned}$$

Thus, if $d \equiv 3 \bmod 4$, then $\gamma_a(T_d) = \frac{2}{5}n$.

Subcase (iii): $d \equiv 0 \bmod 4$.

Then $S = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{3,2}$.

Now, let $S^{'} = S \cup \{(1,1)\}$. It is not hard to see that $S^{'}$ is a global defensive alliance. Also, $S^{'}$ has exactly 3 vertices in each of the components of $< S_{d-1,d-2} >$ and $< S_{d-4i-1,d-4i-2} >$, for $0 \le i \le (d-4)/4$. All we have to show is that any global defensive alliance must contain at least one vertex in $S_{1,0}$. But this is true since the root must be dominated. Therefore, we need at least one more vertex in addition to S, and adding $(1,1)$ (or $(1,2)$) is sufficient to give a global defensive alliance. Thus $S^{'}$ is indeed a global defensive alliance of minimum cardinality. The size of $S^{'}$ is

$$\begin{aligned}
|S^{'}| &= (2^{d-1}+2^{d-2}+2^{d-5}+2^{d-6}+2^{d-9}+2^{d-10}+...+2^{3}+2^{2})+1\\
&= 3(2^{d-2}+2^{d-6}+2^{d-10}+...+2^{2})+1\\
&= 3\left(\frac{2^{2}((2^{4})^{\frac{d}{4}}-1)}{2^{4}-1}\right)+1\\
&= 3\left(\frac{2^{2}(2^{d}-1)}{15}\right)+1=\frac{1}{5}(2^{d+2}-4)+1\\
&= \frac{1}{5}(2(n+1)-4)+1=\frac{2}{5}n+\frac{3}{5}.
\end{aligned}$$

Case 2: $d \equiv 1 \bmod 4$.

Then $S = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{4,3}$.
Let $S^{'} = S \cup \{(1,1),(0,0)\}$.

It is easy to chekc that $S^{'}$ is a global defensive alliance. Also, $S^{'}$ has exactly 3 vertices in each of the components of $< S_{d-1,d-2} >$ and $< S_{d-4i-1,d-4i-2} >$, for $0 \leq i \leq (d-5)/4$. All that remains to show is that any $\gamma_a(T_d)$-set must contain at least two vertices of the set $S_{2,0}$. But this is clear since the root must be dominated and the vertex that dominates the root must have another neighbor in the defensive alliance. Since adding $(1,1)$ (or $(1,2)$) and $(0,0)$ to S gives a global defensive alliance, $S^{'}$ is a $\gamma_a(T_d)$-set. The size of $S^{'}$ is

$$\begin{aligned}
|S^{'}| &= (2^{d-1}+2^{d-2}+2^{d-5}+2^{d-6}+2^{d-9}+2^{d-10}+...+2^{4}+2^{3})+2\\
&= 3(2^{d-2}+2^{d-6}+2^{d-10}+...+2^{3})+2\\
&= 3\left(\frac{2^{3}((2^{4})^{\frac{d-1}{4}}-1)}{2^{4}-1}\right)+2\\
&= \frac{1}{5}(2^{d+2}-8)+2=\frac{1}{5}(2(n+1)-8)+2\\
&= \frac{2}{5}n+\frac{4}{5}.
\end{aligned}$$

Therefore, $\gamma_a(T_d) = \lceil \frac{2}{5}n \rceil$ for all $d \geq 1$. □

Corollary 2.2. *If $d \equiv 2 \bmod 4$ or $d \equiv 3 \bmod 4$ then there is a unique $\gamma_a(T_d)$-set. If $d \equiv 0 \bmod 4$ or $d \equiv 1 \bmod 4$ then there are exactly two $\gamma_a(T_d)$-sets.*

Proof. In case 1, it was shown that each component of $S_{i,i-3}$ had to have at least 3 vertices of any $\gamma_a(T_d)$-set, and there was a unique way to choose exactly 3 vertices. In the subcase (iii), we had two choices to choose a vertex from $S_{1,0}$. Similarly, in case 2 we had to choose two vertices from $S_{2,0}$ and we showed that this can be done in two ways. $\square$

Next, we find the global defensive alliance number of $T_{3,d}$.

Theorem 2.3. *If $d \geq 4$ then $\gamma_a(T_{3,d}) = \lfloor \frac{19}{36} n \rfloor$ if d is odd and $\gamma_a(T_{3,d}) = \lceil \frac{19}{36} n \rceil$ if d is even.*

Proof. Again, define

$$S_{i,j} = \cup_{k=j}^{i} L_k.$$

Consider the set $S_{d,d-2}$. Its induced subgraph, $< S_{d,d-2} >$, is a forest. Each component of the forest is an identical copy of $T_{3,2}$. Given a $\gamma_o(T)$-set S, we claim that each component of $< S_{d,d-2} >$ contains at least 7 vertices of S. Without loss of generality, consider the component C_1 which is the subtree rooted at vertex $(d-2,1)$. First observe that if $v \in S$ is not a leaf, then at least two of its neighbors are also in S. For each child u of $(d-2,1)$ which is not in S, all three children of u must be in S for otherwise S is not a dominating set. Also, if a child of $(d-2,1)$ is in S, at least two of its neighbors must be in S. The children of $(d-2,1)$ have one neighbor in common, $(d-2,1)$. Therefore, if m is the number of children of $(d-2,1)$ which are in S, then we need at least $(2m+1)+3(3-m) = 10-m$ vertices of C_1 to be in S. Therefore, C_1 contains at least 7 vertices of S.

Next, we consider $S_{i,i-3}$, $i \geq 3$. This set induces a graph which is a forest. All the components of $< S_{i,i-3} >$ are identical of copies of the tree $T_{3,3}$. We claim that each component C contains at least 10 vertices of S. Without loss of generality, consider the component C_2 which contains the vertex $(i-3,1)$. First, assume that $(i-3,1)$ is in S. For each child v of the vertex $(i-3,1)$ that is not in S, we need 3 vertices in $S_{i,i-3} \cap T_v$ (where T_v is the subtree rooted at v) to dominate each of v's children. For each child u of $(i-3,1)$ that is in S, we need at least one of u's children, say y, to be in S, and hence at least one child of y to be in S. Therefore, in this case we have that S must contain at least 10 vertices of $S_{i,i-3}$. Second, assume that the vertex $(i-3,1)$ is not in S. For every child u of $(i-3,1)$ that is not in S, one of u's children, say v, must be in S to dominate u, and hence two of v's children must be in S as well. Also, we need at least two vertices to dominate the other two children of u. Therefore, among u's descendants in $S_{i,i-3}$ we need at least 5 vertices in S. Now, we consider the children of

$(i-3,1)$ which are in S. For every child w of $(i-3,1)$ that is in S, at least two of w's children must be in S as well. For each of w's children x that is in S, one of x's children must also be in S. Therefore, among w and its descendants there are at least 5 vertices that must be in S. We conclude that if $(i-3,1)$ is not in S, then S must contain at least 15 vertices in $S_{i,i-3}$.

Therefore, we need at least 10 vertices of S in each component of $S_{i,i-3}$.

Next, we define a set $S^{'}$ which is a global defensive alliance and show that $S^{'}$ contains exactly 7 vertices in all the connected components of $S_{d,d-2}$ and 10 vertices in the components of $S_{d-3-4i,d-6-4i}$, for $0 \leq i \leq \lfloor \frac{d-6}{4} \rfloor$. By what we proved above, this would imply that $S^{'}$ is a $\gamma_a(T_{3,d})$-set.

First let S_1 be the set

$$S_1 = \cup_{i=0}^{\lfloor (d-2)/4 \rfloor} S_{d-4i-1,d-4i-2}$$

and S_2 be the set $S_2 = \{(i,j) : i = d,\ d-3,\ d-4,\ d-7,\ d-8,\ d-11, d-12 ..., d-3-4\lfloor \frac{d-4}{4} \rfloor,\ d-4-4\lfloor \frac{d-4}{4} \rfloor\ ;\ j = 1,\ 4,\ 7,\ 10,\ ...,\ 3^i-2\}$.

We consider two cases.

Case 1: $d \equiv 2 \bmod 4$ or $d \equiv 3 \bmod 4$.

Subcase(i): $d \equiv 2 \bmod 4$.

Then $S_1 = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{1,0}$ and $S_2 = \{(i,j) : i = d,\ d-3,\ d-4,\ d-7,\ d-8,\ ..., 3,\ 2\ ;\ j = 1,\ 4,\ 7,\ 10,\ ...,\ 3^i-2\}$.

Let $S^{'} = S_1 \cup S_2$.

As was shown above, if S is a global defensive alliance then in each component of $S_{d-3-4i,d-6-4i}$, for $0 \leq i \leq \lfloor \frac{d-6}{4} \rfloor$, it must contain at least 10 vertices, and in each component of $S_{d,d-2}$ it must contain at least 7 vertices. It is easy to check that the set $S^{'}$ attains these lower bounds and at the same time it is a global defensive alliance. Therefore, $S^{'}$ is a $\gamma_a(T_{3,d})$-set. The size of $S^{'}$ is

$$\begin{aligned}
|S^{'}| &= 7 \times 3^{d-2} + 10 \times 3^{d-6} + 10 \times 3^{d-10} + ... + 10 \times 3^0 \\
&= 7 \times 3^{d-2} + 10(3^{d-6} + 3^{d-10} + ... + 3^0) \\
&= 7 \times 3^{d-2} + 10 \left(\frac{(3^4)^{\frac{d-2}{4}} - 1}{80} \right) \\
&= \left(7 + \frac{1}{8} \right) 3^{d-2} - \frac{1}{8} = \frac{57}{8} \times \frac{2n}{27} + \frac{57}{8} \times \frac{1}{27} - \frac{1}{8} \\
&= \left\lceil \frac{19}{36} n \right\rceil .
\end{aligned}$$

Subcase (ii): $d \equiv 3 \bmod 4$.

Then $S_1 = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{2,1}$

$S_2 = \{(i,j) : i = d,\ d-3,\ d-4,\ d-7,\ d-8,\ ..., 4,\ 3\,;\ j = 1,\ 4,\ 7,\ 10,\ ...,\ 3^i - 2\}$

Let $S^{'} = S_1 \cup S_2$.

Again, by a similar argument as in subcase (i) it's easy to see that $S^{'}$ is a global defensive alliance with size

$$\begin{aligned}
|S^{'}| &= 7 \times 3^{d-2} + 10 \times 3^{d-6} + 10 \times 3^{d-10} + ... + 10 \times 3^1 \\
&= 7 \times 3^{d-2} + 10(3^{d-6} + 3^{d-10} + ... + 3^1) \\
&= 7 \times 3^{d-2} + 30 \left(\frac{(3^4)^{\frac{d-3}{4}} - 1}{80} \right) \\
&= \frac{57}{8} \times \frac{2n}{27} + \frac{57}{8} \times \frac{1}{27} - \frac{3}{8} \\
&= \left\lfloor \frac{19}{36} n \right\rfloor .
\end{aligned}$$

Case 2: $d \equiv 0 \bmod 4$ or $d \equiv 1 \bmod 4$.

Here we are going to slightly modify the definitions of S_1 and S_2 but, the general argument is the same.

Subcase (i): $d \equiv 0 \bmod 4$.

Then let $S_1 = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{3,2}$ and $S_2 = \{(i,j) : i = d,\ d-3,\ d-4,\ d-7,\ d-8,\ ..., 5,\ 4\,;\ j = 1,\ 4,\ 7,\ 10,\ ...,\ 3^i - 2\}$.

Let $S^{'} = S_1 \cup S_2 \cup \{(1,1)\}$.

$S^{'}$ is a $\gamma_a(T)$-set because we know that we need at least 1 vertex of any global defensive alliance in the subtree formed by the root and its children. The size of $S^{'}$ is

$$\begin{aligned}
|S^{'}| &= 1 + 7 \times 3^{d-2} + 10 \times 3^{d-6} + 10 \times 3^{d-10} + ... + 10 \times 3^2 \\
&= 1 + 7 \times 3^{d-2} + 10(3^{d-6} + 3^{d-10} + ... + 3^2) \\
&= 1 + 7 \times 3^{d-2} + 90 \left(\frac{(3^4)^{\frac{d-4}{4}} - 1}{80} \right) \\
&= 1 + \frac{57}{8} \times \frac{2n}{27} + \frac{57}{8} \times \frac{1}{27} - \frac{9}{8} \\
&= \left\lceil \frac{19}{36} n \right\rceil .
\end{aligned}$$

Subcase (ii): $d \equiv 1 \bmod 4$.

Let $S_1 = S_{d-1,d-2} \cup S_{d-5,d-6} \cup ... \cup S_{4,3}$ and $S_2 = \{(i,j) : i = d,\ d-3,\ d-4,\ d-7,\ d-8,\ ...,6,\ 5\,;\ j = 1,\ 4,\ 7,\ 10,\ ...,\ 3^i - 2\}$.

Let $S^{'} = S_1 \cup S_2 \cup \{(0,1),\ (1,1),\ (2,1)\}$.

It's not hard to see that S is a $\gamma_a(T)$-set since any global defensive alliance needs at least 3 vertices in $S_{2,0}$. The size of S is

$$\begin{aligned} |S| &= 3 + 7 \times 3^{d-2} + 10 \times 3^{d-6} + 10 \times 3^{d-10} + ... + 10 \times 3^3 \\ &= 3 + 7 \times 3^{d-2} + 10(3^{d-6} + 3^{d-10} + ... + 3^3) \\ &= 3 + 7 \times 3^{d-2} + 270 \left(\frac{(3^4)^{\frac{d-5}{4}} - 1}{80} \right) \\ &= 3 + \frac{57}{8} \times \frac{2n}{27} + \frac{57}{8} \times \frac{1}{27} - \frac{27}{8} \\ &= \left\lfloor \frac{19}{36} n \right\rfloor . \end{aligned}$$

□

When k is large, the methods used to prove the above theorems are difficult to apply. The difficulty is in claiming a lower bound that every $\gamma_a(T_{k,d})$-set must have in a given partition of the tree. Therefore, for general k, we give upper and lower bounds for $\gamma_a(T_{k,d})$.

Theorem 2.4. *For $d \geq 2$, and $k \geq 2$,*

$$k^{d-1} \left\lfloor \frac{k-1}{2} \right\rfloor + k^{d-1} + k^{d-2} \leq \gamma_a(T_{k,d}) \leq k^{d-1} \left\lfloor \frac{k-1}{2} \right\rfloor + k^{d-1} + k^{d-2} + k^{d-3}.$$

Proof. To get the lower bound we claim that for any global defensive alliance S, each component C of $< S_{d,d-2} >$ must contain at least $1 + k + k\lfloor \frac{k-1}{2} \rfloor$ vertices of C. The proof is analogous to the case when $k = 3$, which is proven above. Summing over all the components gives the lower bound. For the upper bound, define $S_1 = \cup_{i=0}^{d-1} L_i$ and let S_2 be any set that contains $\lfloor \frac{k-1}{2} \rfloor$ children of each vertex in L_{d-1}. Then $S_1 \cup S_2$ is a global defensive alliance. □

It follows that $\gamma_a(T_{k,d}) \sim k^{d-1} \left\lfloor \frac{k-1}{2} \right\rfloor$, where the asymptotics is taken to be in terms of k. Since the number of vertices of $T_{k,d}$ is $n = \frac{k^{d+1}-1}{k-1}$ we get $\gamma_a(T_{k,d}) \sim \frac{n}{2}$ when k tends to infinity.

Next, we consider the global offensive alliance number of a complete k-ary tree.

3 Offensive Alliances of Complete k-ary trees

It turns out that computing the global offensive alliance number for the complete k-ary tree is significantly easier than computing the global defensive alliance number. We have the following theorem.

Theorem 3.1. *Let $T_{k,d}$ be the complete k-ary tree with depth $d \geq 1$. Then, $\gamma_o(T_{k,d}) = \left\lfloor \frac{n}{k+1} \right\rfloor$.*

Proof. Let $S = \bigcup_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} L_{d-2i-1}$. We consider the following two cases.

Case 1: $d \equiv 1 \bmod 2$.

It is easy to see that S is a global offensive alliance. S is obviously a dominating set, and for every $v \notin S$, v's children and parent are both in S. We prove that any $\gamma_o(T_{k,d})$-set contains at least as many vertices as S.

Let $S^{'}$ be any $\gamma_o(T_{k,d})$-set. Let v be any vertex at depth $d-(2i-1)$, $1 \leq i \leq \frac{d+1}{2}$. Consider the subtree formed by v and its k children. We claim that at least one vertex in this subtree must be in $S^{'}$. Suppose not. Then v's parent must be in $S^{'}$ for otherwise v would not be dominated. Since v is not in $S^{'}$ and its parent is in $S^{'}$ it follows that some of the children of v must be in $S^{'}$, a contradiction. Therefore, every such subtree contains at least one vertex that must be in any $\gamma_o(T_{k,d})$-set. It is easy to see that S contains exactly one vertex for each such subtree. Therefore, S is a minimum cardinality global offensive alliance. The size of S is

$$\begin{aligned} |S| &= k^{d-1} + k^{d-3} + ... + k^2 + k^0 \\ &= \frac{(k^2)^{\frac{d+1}{2}} - 1}{k^2 - 1} \\ &= \frac{k^{d+1} - 1}{k^2 - 1} = \frac{n}{k+1}. \end{aligned}$$

Therefore, if d is odd, $\gamma_o(T_{k,d}) = \frac{n}{k+1}$.

Case 2: $d \equiv 0 \bmod 2$.

As in case 1, it is easy to see that S is a global offensive alliance. Also, by the same argument as in case 1, S is a $\gamma_o(T_{k,d})$-set of size

$$\begin{aligned}
|S| &= k^{d-1}+k^{d-3}+...+k^3+k^1 \\
&= \frac{k((k^2)^{\frac{d}{2}}-1)}{k^2-1} \\
&= \frac{k^{d+1}-k}{(k-1)(k+1)} \\
&= \frac{k^{d+1}-1+1-k}{(k-1)(k+1)} \\
&= \frac{k^{d+1}-1}{(k-1)(k+1)}+\frac{1-k}{(k-1)(k+1)} \\
&= \frac{n}{k+1}-\frac{1}{k+1}.
\end{aligned}$$

Therefore, if d is even, $\gamma_o(T_{k,d}) = \frac{n}{k+1} - \frac{1}{k+1}$.

Thus, $\gamma_o(T_{k,d}) = \lfloor \frac{n}{k+1} \rfloor$ for all $d \geq 1$. □

Note that $\gamma_o(T_{k,d}) \sim \frac{n}{k}$ with respect to k. As k becomes very large the difference between $\gamma_a(T_{k,d})$ and $\gamma_o(T_{k,d})$ approaches to $n/2$. In general, we are interested if this difference can be larger for other trees. This is what we discuss in the next section.

4 Offensive Alliances vs. Defensive Alliances in general trees

In this section, we prove the following theorem.

Theorem 4.1. *For any tree T of order n, $\gamma_a(T) \leq \gamma_o(T) + \frac{n}{2}$.*

Proof. Root the tree T at a vertex of largest *eccentricity*(the eccentricity of a vertex x is equal to $\max_{y \in V(G)} d(x,y)$). Let T have a depth d, and let v be a vertex at depth $d-2$. Let u be v's parent. We are going to proceed by induction on n. We may assume that $diam(T) \geq 3$. Otherwise, T is a star and the theorem holds (this also establishes the base case).

Let T_v be the subtree of T rooted at vertex v. Let $T^{'} = T - T_v$ be the subtree of T obtained by removing all the vertices of T_v, and let $|T^{'}| = n^{'}$. Define P to be the set of children of v in T which are support vertices. Denote by L the set of children of v which are leaves. By assumption on the diameter of T, $|P| \geq 1$. Let y_i denote the number of children of each vertex in P, $1 \leq i \leq |P|$. We first prove the following two claims.

Claim 4.2. $\gamma_o(T^{'}) \leq \gamma_o(T) - |P|$.

Proof. To see this consider a $\gamma_o(T)$-set S. Without loss of generality, we can assume that all the vertices of P are in S (for every $x \in P$ not in S a child of x is in S and we can swap it with x). First, suppose v is not in S. Then v's parent u is dominated by a vertex in $T^{'}$. In this case, it is easy to see that $S - P$ is a global offensive alliance for $T^{'}$. Therefore, in this case $\gamma_o(T^{'}) \leq \gamma_o(T) - |P|$. Next, suppose that v is in S. Then the set $S - P - \{v\} \cup \{u\}$ is global offensive alliance for $T^{'}$. In this case as well, we have the inequality that we desire. ☐

Claim 4.3. *$\gamma_a(T) \leq \gamma_a(T^{'}) + k$, where $k = 1 + |P| + \max\left(\lceil\frac{|L|-|P|}{2}\rceil, 0\right) + \sum_{i=1}^{|P|}\lfloor\frac{y_i-1}{2}\rfloor$.*

Proof. Let $S^{'}$ be a $\gamma_a(T^{'})$-set. Then if we add to $S^{'}$, {v}, the set of vertices in P and at most $\lceil\frac{|L|-|P|}{2}\rceil$ vertices of L, and for each $i \in P$, $\lfloor\frac{y_i-1}{2}\rfloor$ of its children then we get a global defensive alliance for T. A crucial assumption here is that $y_1 \geq 1$. ☐

By Claim 4.3 and the induction hypothesis we have

$$\gamma_a(T) \leq \gamma_a(T^{'}) + k \leq \gamma_o(T^{'}) + \frac{n^{'}}{2} + k.$$

What is left to prove is that $\gamma_o(T^{'}) + \frac{n^{'}}{2} + k \leq \gamma_o(T) + \frac{n}{2}$. By Claim 4.2, it is sufficient to prove that $k - \lfloor\frac{n-n^{'}}{2}\rfloor \leq |P|$. Since $|P| \geq 1$, we have that

$$1+\max\left(\left\lceil\frac{|L|-|P|}{2}\right\rceil, 0\right)+\sum_{i=1}^{|P|}\left\lfloor\frac{y_i-1}{2}\right\rfloor \leq 1+\left\lfloor\frac{|L|}{2}\right\rfloor+\sum_{i=1}^{|P|}\left\lfloor\frac{y_i-1}{2}\right\rfloor \leq \left\lfloor\frac{n-n^{'}}{2}\right\rfloor,$$

as required.

☐

The above bound is best possible. Consider $K_{1,n-1}$ where n is odd. Then $\gamma_o(K_{1,n-1}) = 1$ and $\gamma_a(K_{1,n-1}) = 1 + \frac{n-1}{2}$.

In a connected bipartite graph, each partite set forms a global offensive alliance. It follows that $\gamma_o(T) \leq \frac{n}{2}$ for any tree T. Combining this with the above result we obtain that $|\gamma_a(T) - \gamma_o(T)| \leq \frac{n}{2}$. However, we believe the following stronger result is true.

Question 4.4. *Is it true that for any n-vertex tree T, $\gamma_o(T) \leq \gamma_a(T) + \frac{n}{6}$?*

This conjecture, if true, is essentially best possible due to the following theorem.

Theorem 4.5. *For any constant* $\epsilon > 0$*, there exists a tree* T *with* $\gamma_o(T) \geq \gamma_a(T) + \frac{|V(T)|}{6} - \epsilon$.

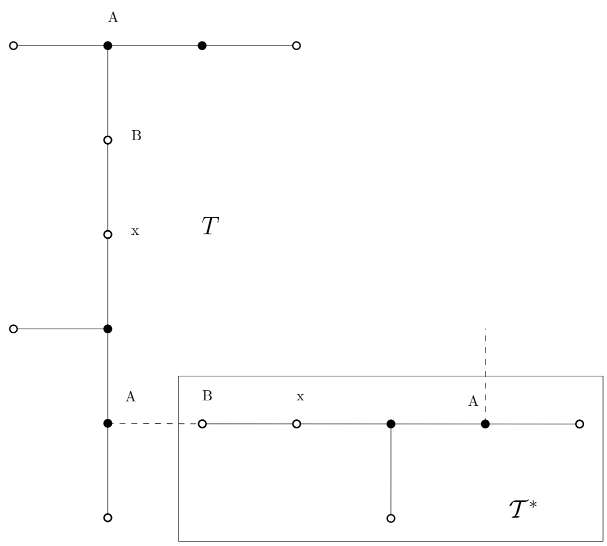

Figure 1: Constructing a tree T satisfying $\gamma_o(T) \geq \gamma_a(T) + \frac{|V(T)|}{6} - \epsilon$.

Proof. The proof is constructive. First observe that for any tree T, we can assume that there exists a $\gamma_o(T)$-set that contains all the *support* vertices of T (i.e. the vertices that are adjacent to leaves). Note that in figure 1, the four black vertices of the tree T form a global defensive alliance in T. By the above observation, we know that the four black vertices are contained in a minimum cardinality global offensive alliance. However, the black vertices do not form a global offensive alliance since the condition is violated for vertices x and B. Therefore, $\gamma_o(T) > \gamma_a(T)$. Now, we append the tree $\mathcal{T}^*$ to T, where we join the vertex A of T with vertex B of $\mathcal{T}^*$. Note that the six black vertices form a global defensive alliance in the new tree, and again they are contained in a minimum global offensive alliance. As before, for every vertex x, we need to add it (or its neighbor B) to the offensive alliance to satisfy the offensive alliance condition. Thus, $\gamma_o(T \cup \mathcal{T}^*) \geq \gamma_a(T \cup \mathcal{T}^*) + 2$. Now, we can repeat the appending operation. We append the tree $\mathcal{T}^*$ to the current tree as before: we join the vertex B of the new tree $\mathcal{T}^*$ to the last added vertex A of the current tree, i.e. the vertex A of the previous $\mathcal{T}^*$. It is easily seen that the difference between the cardinalities of minimum global defensive alliance and minimum global offensive alliance increases by one after each such operation. Since $\mathcal{T}^*$ has 6 vertices, after sufficiently many such operations we will obtain a tree that satisfies the necessary threshold ϵ. This completes the proof. $\square$

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